On a Special Elliptic Ruled Surface of the Ninth Order.

Dissertation.

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By
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Introduction

The object of this paper is to discuss the following problem connected with a tetrahedron.

Are there lines connected with a tetrahedron, T, such that if the vertices are reflected in these lines, the reflections will fall on the opposite faces? If so what is the locus of these lines?

G. T. Bennett* has shown that for a chosen T, there are ∞^1 such lines and that when the opposite pairs of edges of T are equal, the locus consists of 3 cylindroids. First we shall establish the existence of these ∞^1 lines by direct geometric considerations, and then discuss their locus.

* Proc. Lon. Math. Soc. Series 2 Vol. 10. Parts 4 and 5. (1911).

§ 1. Collineation between Vertices and Opposite Faces of a Tetrahedron.

Let $(\alpha x)(a\eta) = 0$, be a collineation between a point x, and a point x'. Then the coefficients of the η's can be taken as the coordinates of the point x'. This gives:

$$\begin{aligned} k x_0' &= a_0(\alpha x) \\ k x_1' &= a_1(\alpha x) \\ k x_2' &= a_2(\alpha x) \\ k x_3' &= a_3(\alpha x) \end{aligned}$$

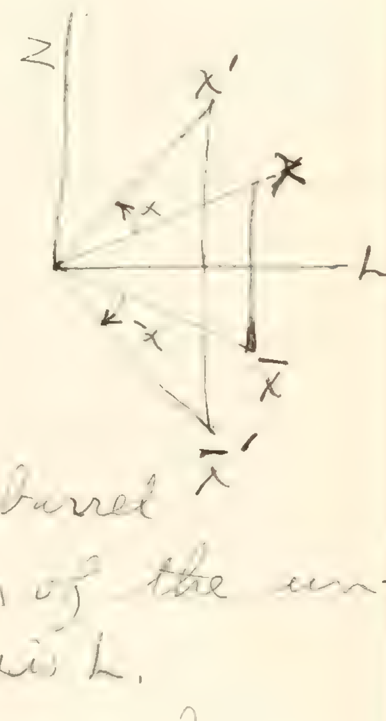
where k is the factor of proportionality. If we let $x_i' = x_i$, the eliminant of these equations will give the fixed points of the collineation. This eliminant is:

$$\begin{vmatrix} a_0\alpha_0-k & a_0\alpha_1 & a_0\alpha_2 & a_0\alpha_3 \\ a_1\alpha_0 & a_1\alpha_1-k & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_0 & a_2\alpha_1 & a_2\alpha_2-k & a_2\alpha_3 \\ a_3\alpha_0 & a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3-k \end{vmatrix} = 0,$$

or $k^4 - f_1 k^3 + f_2 k^2 - f_3 k + f_4 = 0$, where the f's have the following forms

$$\mathcal{I}_1 = (a\alpha) = a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$
$$\mathcal{I}_2 = (\alpha\beta\cdot ab) = |\alpha\beta| \times |ab|.$$
$$\mathcal{I}_3 = (\alpha\beta\gamma\cdot abc) = |\alpha\beta\gamma| \times |abc|$$
$$\mathcal{I}_4 = (\alpha\beta\gamma\delta\cdot abcd) = |\alpha\beta\gamma\delta| \times |abcd|$$

Suppose the above collineation to be a displacement. If this displacement is effected by a screw motion we would have:



$$x' = e^{i\alpha}x$$
$$\bar{x}' = e^{-i\alpha}\bar{x}$$
$$z' = z + p\alpha$$

where p is the pitch of the screw, α the angle turned thru, and the barred variable is the reflection of the unbarred variable in the axis L.

These equations written in homogeneous coordinates are:

$$x_0' = e^{i\alpha}x_0$$
$$x_1' = e^{-i\alpha}x_1$$

$$x_2' = x_2 + p\alpha x_3$$
$$x_3' = x_3$$

$\mathfrak{T}_1$ then is $e^{i\alpha} + e^{-i\alpha} + 1 + 1$

or $\mathfrak{T}_1 = 2 + 2\cos\alpha$.

We shall now ask that this displacement send a vertex of the reference tetrahedron, which we shall designate as T, onto the face opposite. If the vertex $(1,0,0,0)$, is to go onto the opposite face, $(0,1,1,1)$ then $(\alpha x) = \alpha_0$, and $(u\eta) = u_0$ and consequently $(u)(\eta) = 0$ becomes $a_0\alpha_0 = 0$. In the same way, we get for the displacements that send the three other vertices onto the corresponding opposite faces, —

$$a_1\alpha_1 = 0, \quad a_2\alpha_2 = 0, \quad a_3\alpha_3 = 0$$

If all the vertices are required to go onto the opposite faces simultaneously, we shall have

* $\eta \equiv (a\alpha) = 0$.

* The converse is easily seen to be true, i.e. if $\eta_1 = 0$, the corresponding displacement sends the vertices on to the opposite faces.

For the screw displacement above, where $T_1 = 2 + 2\cos\alpha$, we see that $T_1 = 0$ requires $\alpha = 180°$. Therefore from the above considerations we deduce this statement:

There are displacements of T which send each vertex onto the opposite face, and these displacements consist of rotations about an axis thru an angle of $180°$, plus a translation.

This statement is also substantiated by the constants involved. The general collineation has 16 coefficients and therefore involves 15 independent constants. In the displacement considered, we must leave unaltered the plane at infinity, the absolute in this plane, and the size of T. This is 3 + 5 + 1 or 9 conditions. It is one condition on each vertex that

it lie on the face opposite it, or 4 in all. This leaves 15-9-4 or 2 constants at our disposal, — a number just sufficient to determine the ~~∞²~~ axes of the rotations.

Instead of thinking of a single tetrahedron, T, and its positions after the displacements, we shall call the displaced tetrahedra T', and the original tetrahedron, T, where T is to be the reference tetrahedron throughout this paper. Then T is inscribed to each T', and T and T' are such tetrahedra that a rotation of T thru angle of 180° plus a translation would make T and T' coincident. The pitch of this screw motion will now be taken to be 0, and then T' is a reflection of T in the ~~∞~~ axis of rotation.

§2. The Order of Rotations.

Let us choose Γ to be the tetrahedron of reference, and consider the 4 reflections.

R_0: $x_0' = -x_0$, $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3$

R_1: $x_0' = x_0$, $x_1' = -x_1$, $x_2' = x_2$, $x_3' = x_3$

R_2: $x_0' = x_0$, $x_1' = x_1$, $x_2' = -x_2$, $x_3' = x_3$

R_3: $x_0' = x_0$, $x_1' = x_1$, $x_2' = x_2$, $x_3' = -x_3$

R_0 sends a line p, whose coordinates are

$$p_{01}, p_{02}, p_{03}, p_{12}, p_{23}, p_{31}.$$

into a line whose coordinates are

$$-p_{01}, -p_{02}, -p_{03}, p_{12}, p_{23}, p_{31}.$$

We shall call this new line pR_0

In the same way R_1 sends the line p into pR_1 with coordinates

$$-p_{01}, p_{02}, p_{03}, -p_{12}, p_{23}, -p_{31}.$$

If π be a line with the coordinates π_{ij}; and π meets pR_0, then

(1) $-p_{01}\pi_{01} - p_{02}\pi_{02} - p_{03}\pi_{03} + p_{23}\pi_{23} + p_{31}\pi_{31}$
$+ p_{12}\pi_{12} = 0.$

If π meets pR_1, then

(2) $-p_{01}\pi_{01} + p_{02}\pi_{02} + p_{03}\pi_{03} + p_{23}\pi_{23} + p_{31}\pi_{31} - p_{12}\pi_{12}$ [illegible] we get a the condition for π to meet both pR_0 and pR_1 i.e.

(3) $p_{01}\pi_{01} = p_{23}\pi_{23}$, a linear complex.

In the same way, we get as the condition for π to meet both pR_0 and pR_2

(4) $p_{02}\pi_{02} = p_{31}\pi_{31}$, a 2nd linear complex.

and for π to meet both pR_0 and pR_3

(5) $p_{03}\pi_{03} = p_{12}\pi_{12}$, a 3rd linear complex.

<u>As π is then common to 3 linear complexes, there are ∞^1 such lines, π.</u>

If π then meets the 4 reflections pR_0, pR_1, pR_2, and pR_3, and there are ∞^1 such lines π. [illegible] pR_i [illegible] This line p also meets the lines πR_i, as seen by observing that p and π play dual roles

Next let us get the equation of the quadric on which p_{K_i} lie. For any given line π, we have from equations (3), (4), and (5) above:

$$(x_0 y_1 - x_1 y_0)\pi_{01} - (x_2 y_3 - x_3 y_2)\pi_{23} = 0$$

$$(x_0 y_2 - x_2 y_0)\pi_{02} - (x_3 y_1 - x_1 y_3)\pi_{31} = 0$$

$$(x_0 y_3 - x_3 y_0)\pi_{03} - (x_1 y_2 - x_2 y_1)\pi_{12} = 0$$

Eliminate y_1, y_2 and y_3. This gives

$$\rho y_0 = x_0 (x_0^2 \pi_{01}\pi_{02}\pi_{03} + x_1^2 \pi_{12}\pi_{13}\pi_{10} + x_2^2 \pi_{20}\pi_{21}\pi_{23} + x_3^2 \pi_{31}\pi_{32}\pi_{30})$$

Similarly $\rho y_1 = -x_1 (\quad " \quad)$

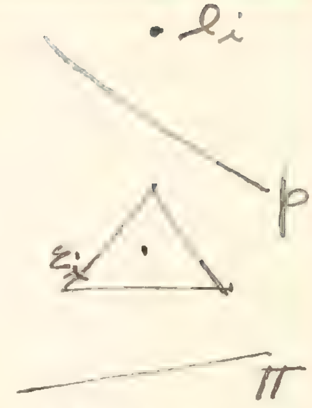
$\rho y_2 = -x_2 (\quad " \quad)$

$\rho y_3 = -x_3 (\quad " \quad)$

These equations are satisfied by $x \equiv y$ and also by:

(2) $x_0^2 \pi_{01}\pi_{02}\pi_{03} + x_1^2 \pi_{10}\pi_{12}\pi_{13} + x_2^2 \pi_{20}\pi_{21}\pi_{23} + x_3^2 \pi_{30}\pi_{31}\pi_{32} = 0$.

the only square terms appear. This is a quadric referred to a self-polar tetrahedron of reference, namely π.



The relation of p and π in (3), (4) and (5) is, that the reflection of l_i, in p and π, is on E_i, where l_i is a vertex, and is the pole of plane, of T. And if the reflection set up by p and π send T into an inscribed tetrahedron, T', then p and π are on a quadric apolar to T.

Let us now make our reflection a physical one by sending the line π to infinity in a direction perpendicular to p. [illegible] makes π the polar line, as to the absolute, of the point where p meets the plane at infinity. Thus the quadric, touching the plane at infinity, is a paraboloid, and becomes, if the [illegible] π were taken to infinity, [illegible] orthic.*

So at infinity, we have for any one such

* The word orthic is used here in the perpendicular sense, defined by the lines p and π in this paragraph.

quadric, a point of contact with the plane at infinity, which we shall call p, and a line π, which is the polar line of p, with regard to the absolute.

$\left(\frac{x^2}{y}\right) \equiv \frac{x_0^2}{y_0} + \frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} = 0$, represents any quadric referred to a given self-polar tetrahedron as tetrahedron of reference. To make this quadric a paraboloid it must be required to touch the plane at infinity, which is $(x) = 0$. The condition for this is:

$$\frac{1}{y_1 y_2 y_3} + \frac{1}{y_0 y_2 y_3} + \frac{1}{y_0 y_1 y_3} + \frac{1}{y_0 y_1 y_2} = 0$$

or $(y) = y_0 + y_1 + y_2 + y_3 = 0$.

Then $\left(\frac{x^2}{y}\right) = 0$ is to be orthic, i.e. must be apolar to the absolute. As will be shown later, the equation of the absolute can be written as $(A\xi)^2 = 0$*. For $\left(\frac{x^2}{y}\right) = 0$ to be apolar to $(A\xi)^2 = 0$, the necessary condition is:

$$\left(\frac{A^2}{y}\right) \equiv \frac{A_{00}}{y_0} + \frac{A_{11}}{y_1} + \frac{A_{22}}{y_2} + \frac{A_{33}}{y_3} = 0.$$

*where A_{ii} is the square of the area of the face of T which is opposite the vertex x_i.

Then all lines, p, lie on the quadrics represented by $\left(\frac{X^2}{y}\right) = 0$, where these quadrics are subjected to the two relations $(y) = 0$ and $\left(\frac{A^2}{y}\right) = 0$. Evidently from the fore-going statements the axis of any one of these quadrics is on y, and as p is orthic with regard to π and y is on π, it can be seen that p is one of the principal generators of the quadric. Equally well can the other principal generator be seen to be a line, p', for which the corresponding π' is on y and the former p. i.e.

<u>all the lines sought, are the principal generators of the quadrics</u> $\left(\frac{X^2}{y}\right) = 0$, <u>where</u> $(y) = 0$ <u>and</u> $\left(\frac{A^2}{y}\right) = 0$.

A case in point is the rectangular hyperbolic parab[illegible], $xy = z$. The polar plane of a point, $x'y'z'$ is $xy' + x'y = z + z'$.

This equation is satisfied by

(1) $x', -y', -z'$, which is the reflection of x', y', z' in the line $x = 0$.

(2) $-x', y', -z'$, which is the reflection of x', y', z' in the y-axis.

These two lines are the principal

generators of $xy = z$. If $(x'y'z')$ be a vertex of T_1 and $xy' + x'y = z + z'$ be the opposite face of T_1 then a reflection of this vertex in either of the above lines is a point of the face opposite this vertex. Thus if we choose three points (a,b,c), (d,e,f) and (r,s,t) and get their polar planes as to $xy = zz$, they in turn will determine the fourth point which, with the given three points, determine a tetrahedron which is self polar with regard to $xy = zz$. The plane on the above points is

$$(7)\quad \begin{vmatrix} x & y & z & 1 \\ a & b & c & 1 \\ d & e & f & 1 \\ r & s & t & 1 \end{vmatrix} = 0 \quad \text{or} \quad Ax - By + Cz - D = 0$$

where A is the first minor of x, B of y, etc.

The three reflected points $(a,-b,-c)$, $(d,-e,-f)$, $(r,-s,-t)$ are on the line

$$(8)\quad Ax + By - Cz - D = 0.$$

faces of T_1 in collinear points.

and the three reflections in the x-axis are on the plane

(9) $Ax + By + Cz + D = 0$.

The polar point of the plane (7) with regard to $xy = 2z$ is the point $\left(\frac{B}{C}, -\frac{A}{C}, -\frac{D}{C}\right)$ which is on the planes (8) and (9) as seen by substituting these values for (x, y, z) in their equations. Similarly the polar point of (8) is $\left(\frac{B}{C}, \frac{A}{C}, \frac{D}{C}\right)$ which is on (7) and the polar point of (9) is $\left(-\frac{B}{C}, -\frac{A}{C}, \frac{D}{C}\right)$ which is also on (7).

The polar planes of $(a, -b, -c)$, $(d, -e, -f)$ and $(r, -s, -t)$ with regard to $xy = 2z$ are respectively:

(1') $bx - ay + z - c = 0$

(2') $ex - dy + z - f = 0$

(3') $sx - ry + z - t = 0$.

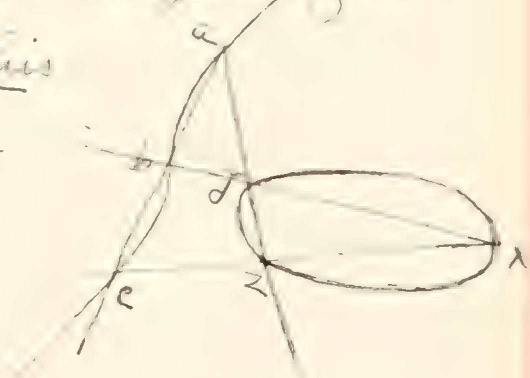
$(d, -e, -f)$ is on (1') and $(a, -b, -c)$ on (2') if $bd + ae - f - c = 0$. $(r, -s, -t)$ is on (1') and $(a, -b, -c)$ is on (3') if $br + as - t - c = 0$. And

$(r, -s, -t)$ is on $(2')$ and $(d, -e, -f)$ on $(3')$ if $er + ds - f - t = 0$. These three conditions make $(a, -b, -c)$, $(d, -e, -f)$, $(r, -s, -t)$ and $\left(\frac{B}{C}, \frac{A}{C}, \frac{D}{C}\right)$ the four vertices of a self-polar tetrahedron, T. The same conditions make the reflections of these four vertices, in the x-axis, lie upon the faces of T and form a tetrahedron, T', to which T is circumscribed. The same conditions make the reflections in the y-axis form the tetrahedron, T'', to which T is also circumscribed.

As there are only 3 conditions necessary to determine T' or T'', it is at once seen that there are ∞^1 lines about which T can be rotated thru $180°$ and thus become coincident with the T''s.

§3. Introduction of Elliptic Functions.

$\left(\frac{x^2}{y}\right) = 0$, is a cubic surface containing the 6 edges of T. Consequently, when we make $(y) = 0$, and so get the intersection of this surface and the plane at infinity, we have a cubic curve which passes thru the 6 points where the 6 edges of T meet this plane. This enables us to introduce an elliptic parameter in terms of which we can express this cubic as well as other curves arising from later considerations.



Let the above 6 points be named a, b, c, x, y, z, as in the figure.

Then since these are elliptic parameters of the cubic and are by 3's on a line,

$$a+b+c=0, \quad b+x+y=0,$$
$$a+y+z=0, \quad c+x+z=0.$$

Adding the last three equations and substituting the first in the result, we have,

$x+y+z=0$, or $=\omega_1$ where ω_1 is a half-period. Then

$$x=\omega_1-y-z=\omega_1+a$$
$$y=\omega_1+b, \qquad z=\omega_1+c.$$

Consider $y_i=h_i\,\sigma(u-a)\,\sigma(u-b)\,\sigma(u-c)$
This represents a line picking out the three values a, b, and c, whose sum is 0.

The line on a, b, c, is $y_0=h_0\,\sigma(u-a)\,\sigma(u-b)\,\sigma(u-c)$

" on a, y, z, " $y_1=h_1\,\sigma(u-a)\,\sigma(u-b-\omega_1)\,\sigma(u-c-\omega_1)$

and similar equations for lines on b, y, z and c, x, z respectively.

The function $\sigma(u-b-\omega_1)=\sigma_1(u-b)$ where σ_1 is the allied σ function.

The above equations then become

$y_0 = \lambda_0 \sigma(u-a)\sigma(u-b)\sigma(u-c)$

$y_1 = \lambda_1 \sigma(u-a)\sigma_1(u-b)\sigma_1(u-c)$, etc.

in which λ is to be so determined that $(y) = 0$, and $\left(\frac{\lambda^2}{y}\right) = 0$.

For $(y) = 0$ we have

$\lambda_0 \sigma(u-a)\sigma(u-b)\sigma(u-c) + \lambda_1 \sigma(u-a)\sigma_1(u-b)$
$\sigma_1(u-c) + \lambda_2 \sigma_1(u-a)\sigma(u-b)\sigma_1(u-c) + \lambda_3(\text{-------}) = 0.$

Dividing by the coefficient of λ_0 we get

(10) $\lambda_0 + \lambda_1 \dfrac{\sigma_1(u-b)\sigma_1(u-c)}{\sigma(u-b)\sigma(u-c)} + \lambda_2(\text{----}) + \lambda_3(\text{----}) = 0.$

We shall now remove the infinities for a, b and c, by writing $u=a$, $u=b$, $u=c$ in turn in the above equation. (σ_1 in this case $=1$, as $\sigma_1(0) = \sigma(-\omega_1) = 1$.)

This gives

$$\left[\lambda_2 \frac{\sigma_1(a-c)}{\sigma(a-c)} + \lambda_3 \frac{\sigma_1(a-b)}{\sigma(a-b)}\right]\frac{1}{u-a} = 0$$

$$\left[\lambda_1 \frac{\sigma_1(b-c)}{\sigma(b-c)} + \lambda_3 \frac{\sigma_1(a-b)}{\sigma(a-b)}\right]\frac{1}{u-b} = 0$$

$$\left[\lambda_1 \frac{\sigma_1(b-c)}{\sigma(b-c)} + \lambda_2 \frac{\sigma_1(a-c)}{\sigma(a-c)}\right]\frac{1}{u-c} = 0$$

and eliminating the factors $\frac{1}{u-a}$ etc. we remove the values for which these

expressions would become infinite.

Solving for λ, we get

$$\lambda_1 \frac{\sigma_1(b-c)}{\sigma(b-c)} = 0 \qquad \text{or} \qquad \lambda_1 = \frac{\sigma(b-c)}{\sigma_1(b-c)}$$

$$\lambda_2 \frac{\sigma_1(a-c)}{\sigma(a-c)} = 0 \qquad \qquad \lambda_2 = \frac{\sigma(a-c)}{\sigma_1(a-c)}$$

$$\lambda_3 \frac{\sigma_1(a-b)}{\sigma(a-b)} = 0 \qquad \qquad \lambda_3 = \frac{\sigma(a-b)}{\sigma_1(a-b)}$$

Substituting these values in (10) we have

$$(11) \quad \lambda_0 + \frac{\sigma(b-c)\sigma_1(u-b)\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)} + \cdots + \cdots = 0.$$

To determine λ_0, let $u = a + \omega_1$

Then $u - a = \omega_1$ and $\sigma_1(u-a) = \sigma_1\omega_1 = 0$,

thus eliminating the last two terms in equation (11) and leaving

$$(12) \quad \lambda_0 + \frac{\sigma(b-c)\sigma_1(u-b)\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)} = 0$$

Now

$$\frac{\sigma_1(a-b)}{\sigma(a-b)} = \sqrt{p(a-b) - e_1}$$

$$\frac{\sigma_1(a-b+\omega_1)}{\sigma(a-b+\omega_1)} = \sqrt{p(a-b+\omega_1) - e_1}$$

$$\frac{\sigma_1(a-b)}{\sigma(a-b)} \cdot \frac{\sigma_1(a-b+\omega_1)}{\sigma(a-b+\omega_1)} = \sqrt{[p(a-b)-e_1][p(a-b+\omega_1)-e_1]}$$

$$= \sqrt{(e_3-e_1)(e_2-e_1)}$$

$$\frac{\sigma_1(u-b)}{\sigma(u-b)} = \frac{\sigma_1(a-b+\omega_1)}{\sigma(a-b+\omega_1)} \quad \text{for } u = a+\omega_1$$
etc.

und as
$$\frac{\sigma_1(u-b+\omega_1)}{\sigma(u-b+\omega_1)} = \sqrt{(e_1-e_2)(e_1-e_3)}\,\frac{\sigma(a-b)}{\sigma_1(a-b)}$$

$$\frac{\sigma_1(u-b)}{\sigma(u-b)} = \frac{\sigma(a-b)}{\sigma_1(a-b)}\sqrt{(e_1-e_2)(e_1-e_3)}$$

Similarly
$$\frac{\sigma_1(u-c)}{\sigma(u-c)} = \frac{\sigma(a-c)}{\sigma_1(a-c)}\sqrt{(e_1-e_2)(e_1-e_3)}$$

Substituting these values in (12), we get:
$$\lambda_0 = (e_1-e_2)(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)}$$

Therefore (11) becomes:

(13) $$(e_1-e_2)(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)}$$
$$+\sum^{3} \frac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)} = 0$$

This equation is the one corresponding to $(y) = 0$, and so

$$(14)\begin{cases} y_0 = \dfrac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)}\,(e_1-e_2)(e_1-e_3) \\[2ex] y_1 = \dfrac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)} \end{cases}$$

and similar expressions for y_2 and y_3.

In the last three terms of this equation, let us substitute $u + \omega_1$ for u. Since

$$\frac{\sigma_1(u-b)}{\sigma(u-b)} = \frac{\sigma(u-b)}{\sigma_1(u-b)}\sqrt{(e_1-e_3)(e_1-e_2)}$$

this gives

$$(15)\quad (e_1-e_2)(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)}$$

$$+\sum^{3}\frac{\sigma(b-c)}{\sigma_1(b-c)}\sqrt{(e_1-e_2)(e_1-e_3)}\,\frac{\sigma(u-b)}{\sigma_1(u-b)}$$

$$\sqrt{(e_1-e_3)(e_1-e_3)}\;\frac{\sigma(u-c)}{\sigma_1(u-c)} = 0$$

and dividing by $(e_1-e_2)(e_1-e_3)$

$$(15)\quad \frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)} + \sum^{3}\frac{\sigma(b-c)\,\sigma(u-b)\,\sigma(u-c)}{\sigma_1(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)} = 0$$

By a proper substitution of the values of the y's of (14) in this equation, the result is;

$$\frac{\sigma^2(b-c)\,\sigma^2(c-a)\,\sigma^2(a-b)}{\sigma_1^2(b-c)\,\sigma_1^2(c-a)\,\sigma_1^2(a-b)}(l_1-l_2)(l_1-l_3)\,\frac{1}{y_0}$$

$$+\sum^{3}\frac{\sigma^2(b-c)}{\sigma_1^2(b-c)}\,\frac{1}{y_1}=0$$

This is manifestly of the form

$$\left(\frac{A^2}{y}\right)\equiv\frac{A_{00}}{y_0}+\frac{A_{11}}{y_1}+\frac{A_{22}}{y_2}+\frac{A_{33}}{y_3}=0,$$

Where

(17)
$$\begin{cases} A_{00}=(l_1-l_2)(l_1-l_3)\,\dfrac{\sigma^2(b-c)\,\sigma^2(c-a)\,\sigma^2(a-b)}{\sigma_1^2(b-c)\,\sigma_1^2(c-a)\,\sigma_1^2(a-b)} \\ A_{11}=\dfrac{\sigma^2(b-c)}{\sigma_1^2(b-c)} \\ A_{22}=\dfrac{\sigma^2(c-a)}{\sigma_1^2(c-a)} \\ A_{33}=\dfrac{\sigma^2(a-b)}{\sigma_1^2(a-b)} \\ A_{00}=A_{11}A_{22}A_{33}(l_1-l_2)(l_1-l_3) \end{cases}$$

and $y_0^2=A_{00}(l_1-l_2)(l_1-l_3)$

§4. The Plane at Infinity

The reflection of a vertex, V_i of Π in any line at infinity will lie on the face, F_i, of Π. For, to get a reflection of a point in two lines we take the fourth harmonic of this point and the two intersections of the two given lines with the line that can be drawn thru the given point and the two lines. Now if we take π as any line in W,* and p' is its polar point as to the absolute, then any line on p' will do for the second of the lines of reflection. In particular we can take the line on p' that passes thru the intersection of π and F_i. Then the reflection of V_i and

* Hereafter we shall call the plane at infinity, W.

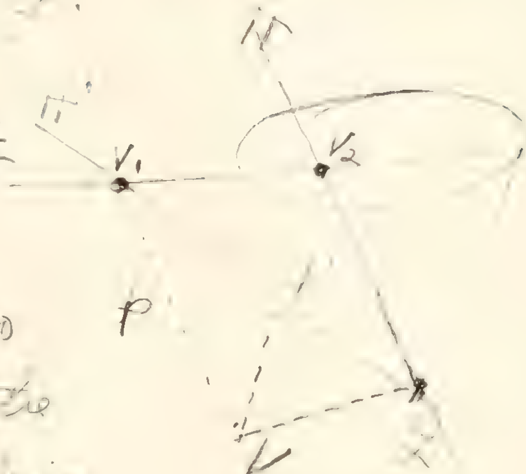
be at this intersection and consequently would lie on F_2. The entire plane at infinity is then to be included in the line locus, which we shall simply call Ω. Some of the lines in W though, are in a special way, lines of Ω, as we shall now see.

First let us find the locus in W of the lines π, as defined in §2. For each π, there is a point y on it, and for all the quadrics which are orthic paraboloids and self polar with regard to T, the corresponding y's lie on the cubic curve, $\left(\frac{A^2}{y}\right)=0$. We shall now get the locus of all the lines π, corresponding to the y's.

The two principal generators of any of these quadrics are perpendicular to each other and to

the axis of the same paraboloid, which is [illegible] p.

So in W, we have this geometrical relation:



p, one of the two generators from V, the vertex of the quadric, meets W in a point V_1 which is the polar point of π as to the absolute. Because of the perpendicularity mentioned, the point V_2, where the other generator than V_1 meets W, will with V_1 have it be on π. And if p is the generator on V_2, then π will be the line on V_1. [illegible]

Regarding the point-pairs,* that arise from the principal generators of the quadrics here considered, it has already been stated that they are on two lines, π and π', that meet on the cubic curve, $\left(\frac{A^2}{y}\right)=0$. These line pairs are conjugates as to the absolute and form the degenerate conics of the net of apolar conics of ~~the~~ some cubic ~~absolute~~. The cubic, $\left(\frac{A^2}{y}\right)$, is then the Jacobian of this net and the Hessian of the cubic(s) of which this net is the polar net.* The Cayleyan of the latter cubic would be enveloped by these line pairs, and therefore the reciprocal of this Cayleyan, as to the absolute, will pass thru the point-pairs, as V_1 and V_2. As lines of Ω pass thru these points, this cubic will be at least a part

* Clifford's Papers," Polar Theory of the Plane Cubic.

of the intersection of Ω and W.

The complete intersection is a 9-ic, as will be shown later and there is yet a sextic to account for. This sextic consists of the 6 tangents to the above cubic at its intersections with the absolute. These 6 lines, p, will be paired off with the 6 lines, π, that are tangents to the absolute at these intersections. The p's that intersect W at these intersections will be perpendicular to the above 6 lines, p, and thus we see that these 6 points are the intersections of W and the sextic double curve of S_6.

The plane 9-ic cut out of Ω by any plane, will be an elliptic 9-ic and consequently must have 27 double points. In the plane W, we have these double points -

evidence. They are the 15 meets of the 6 lines; the 6 points where the cubic meets the absolute; and the 6 points where the tangents of the cubic (at its intersection with the absolute) again meet the cubic.

Since the cubic $\left(\frac{A^2}{\gamma}\right)$ is the Hessian of, and the cubic above, is the reciprocal of the Cayleyan of the coefficients of the same cubic, this reciprocal cubic should be expressible in terms of the coefficients of $\left(\frac{A^2}{\gamma}\right) = 0$, i.e. in terms of the areas of the faces and the lengths of the edges of T.

§ 5. Special Lines of Ω.

Bennett* has mentioned the following lines.

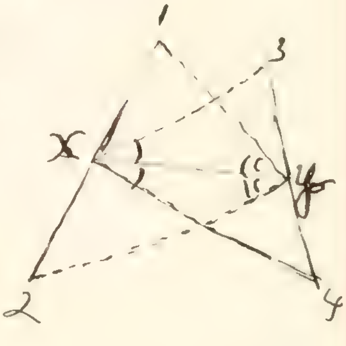
(1) The two perpendicular normals to an edge at its middle point, each equally inclined to the faces through the edge; i.e. lying in the planes that bisect the dihedral angles on an edge.

(2) Any line which can be drawn meeting two opposite edges and bisecting, internally or externally, at each of its extremities, the angle subtended by the opposite edge.

It is seen from simple geometric observations that these lines are proper lines of Ω. The lines (1) give a double-point of the surface at each mid-edge of T', and as the generators that meet in these points are

* Proc. Lon. Math. Soc. ser. [illegible] Vol. 10 Part [illegible]

perpendicular pairs, these points are on the double-curve (C).



For each pair of opposite edges of T', there are seven such lines as (2). This is seen from the correspondence set up. For if $\overline{12}$ and $\overline{34}$ are opposite edges of T', then for two points 1 and 2 on one line, we have one y on the other line and this point y, determines a definite point x on the line on 1 and 2. This x, in turn, determines 2 points 3, and 4 on the 2nd line. So our correspondence is of such a nature that 2 x's pick out 2 y's and y's pick out 2 x's, i.e. the correspondence is 4-to-4, and therefore has 4+4 or 8 coincidences. Thus there are 8 lines of the type (2). Since the plane

at infinity is to be factored out in the consideration of Ω, the line at infinity in the two class must not be counted. So there are 7 such lines of Ω on each edge of the tetrahedron of reference. These with the pair of lines in each edge make a total of 9 lines on each edge of T. There are no more than 1 of these lines.

This shows that the degree of Ω is 3 since a line meets Ω in 3 points. Moreover, we can write the equation of Ω from this view-point, i.e. we can express the coordinates of any line of Ω in terms of an elliptic parameter.

To do this, we will suppose that the intersection cubic of Ω and W is expressed in terms of an elliptic

parameter, u. Then since all lines of Ω meet this cubic, any one of these lines can be named in terms of the same parameter u, there being factors like $\sigma(u-a)$ for each line that meets an edge of the tetrahedron of reference, and so we have

$$p_{ij} = \sigma(u-a_{ij})\,\sigma(u-b_{ij}) \cdots \sigma(u-k_{ij})$$

For the pair of perpendicular lines at the mid-points of an edge, the parameters would differ only by a period. i.e. if one is $u-\alpha$, the other is $(u-\alpha-\omega)$. Then we would have

$$p_{ij} = \sigma(u-a_{ij})\,\sigma(u-b_{ij})\,\sigma(u-c_{ij})$$
$$\sigma(u-d_{ij})\,\sigma(u-e_{ij})\,\sigma(u-f_{ij})$$
$$\sigma(u-g_{ij})\,\sigma(u-h_{ij})\,\sigma(u-h_{ij}-\omega)$$

Moreover, since the 1st 7 of these factors represent the 7 lines that meet the edge ij [illegible], i.e. the

factors of p_{ij} that make it meet the edge $(i\,j)$ will equally well make the same line meet the edge $k\,l$.

~~So~~ if $p_{01} = \sigma(u-a_{01}) \cdots\cdots \sigma(u-g_{01})\sigma(u-h_{01})\sigma(u-h_{01}-\omega)$

~~then~~ $p_{23} = \sigma(u-a_{01}) \cdots\cdots \sigma(u-g_{01})\,\sigma(u-h_{23})\sigma(u-h_{23}-\omega)$

and similarily for p_{02}, p_{13}, and p_{03}, p_{12}

§. 6 The Double Curve

It has already been stated that the lines of the surface Ω, pair off in such a way that the lines of each pair are the principal generators of an orthic paraboloid. Therefore the surface Ω has a double point curve on it, - the intersection of the lines in each pair, - which is the locus of the vertices of all the orthic paraboloids of the system, i.e. of all such quadrics that have T as their tetrahedron of reference, where T is self-polar. In each line of Ω is a tangent plane of Ω, and so the tangent planes of each pair of lines coincide in the plane tangent to the one of the quadrics, (at its vertex,) of which the lines of the pair are the principal generators. Then, consequently,

there will be a double-plane curve which these planes will envelop. We shall now proceed to find the equations of these two double curves, getting the double curve in planes first.

The points y and the lines, cut out in N by the tangent plane at the, of §2, are pole and polar as to the absolute and it has already been stated that the two principal generators of the paraboloid that touches the plane at infinity in this point y, meet π in points that are conjugates as to the absolute. So to get the plane that is tangent to the paraboloid at the vertex, we shall take the polar plane of y as to the circumsphere of π, and then take the plane parallel to this that touches the quadric.

The circum-sphere is given in quadri-planar coordinates as:

$$\frac{l_{21}^2 \bar{x}_1 \bar{x}_2}{x_1' x_2'} + \frac{l_{20}^2 \bar{x}_2 \bar{x}_0}{x_2' x_0'} + \frac{l_{01}^2 \bar{x}_0 \bar{x}_1}{x_0' x_1'} + \frac{l_{03}^2 \bar{x}_0 \bar{x}_3}{x_0' x_3'} + \frac{l_{13}^2 \bar{x}_1 \bar{x}_3}{x_1' x_3'} + \frac{l_{23}^2 \bar{x}_2 \bar{x}_3}{x_2' x_3'} = 0 \quad *$$

where l_{ij} is an edge of π, and x_0', x_1', x_2', x_3' are the altitudes of π, $\bar{x}_i$

* Roger's revision of Salmon's Geom. of 3 dimensions, p 235

the corresponding faces $x_0=0$, $x_1=0$, etc.
* in Barycentric projective coordinates, this is easily seen to be:

$$l_{21}^2 x_2 x_1 + l_{20}^2 x_2 x_0 + l_{01}^2 x_0 x_1 + l_{03}^2 x_0 x_3$$
$$+ l_{13}^2 x_1 x_3 + l_{23}^2 x_2 x_3 = 0$$

or $\sum^6 l_{ij}^2 x_{ij} = 0$. $(i = 0,1,2,3,\ j = 0,1,2,3)$

The polar plane of y, as to this sphere is: $\sum^6 l_{ij}^2 (x_i y_j + x_j y_i) = 0$

which can be written

$$\sum^6 (l_{ij}^2 y_j + l_{ik}^2 y_k + l_{im}^2 y_m) x_i = 0 \text{ or } (cx) = 0$$

(where $j \neq k \neq m \neq i$)

Any plane parallel to $(cx) = 0$ would be

(17') represented by: $(c_0+\lambda)x_0 + (c_1+\lambda)x_1 + (c_2+\lambda)x_2 + (c_3+\lambda)x_3 = 0.$

The condition for this plane to touch the quadric $\left(\frac{x^2}{y}\right) = 0$, is:

$$\frac{(c_0+\lambda)^2}{y_1 y_2 y_3} + \frac{(c_1+\lambda)^2}{y_0 y_2 y_3} + \frac{(c_2+\lambda)^2}{y_0 y_1 y_3} + \frac{(c_3+\lambda)^2}{y_0 y_1 y_2} = 0$$

or $(c_0+\lambda)^2 y_0 + (c_1+\lambda)^2 y_1 + (c_2+\lambda)^2 y_2 + (c_3+\lambda)^2 y_3 = 0$

Which in terms of λ is:

$\lambda^2(y) + 2\lambda(cy) + (c^2 y) = 0.$

The y's, we recall, are to satisfy the two relations $(y) = 0$, and $\left(\frac{A^2}{y}\right) = 0$.

$(c^2 y)$ consists (when the values of c are substituted) of 4 terms like:

$y_1 y_2 y_3 \left[2(l_{12}^2 l_{23}^2 + l_{23}^2 l_{31}^2 + l_{31}^2 l_{12}^2) - l_{12}^4 - l_{13}^4 - l_{23}^4\right]$

and it is seen at once that this coefficient of $y_1 y_2 y_3$ is $16 A_{00}$.

Then $(c^2 y) = 16\, y_0 y_1 y_2 y_3 \left(\frac{A^2}{y}\right)$ and consequently $= 0$.

The above equation in λ then reduces to: $2\lambda(cy) = 0$ which has the roots $\lambda = 0, \infty$. $\lambda = \infty$ substituted in equation (17') gives $(x) = 0$, the plane at infinity. $\lambda = 0$, gives the plane $(cx) = 0$ which then must be the tangent plane at the vertex. and as y traces out the cubic curve, $\left(\frac{A^2}{y}\right) = 0$, in the plane

at infinity, $(\mathfrak{C}X) = 0$ envelops the vertices of the paraboloids corresponding to the y's.

The equation of this double curve (in planes) is given at once by substituting for the y's the values given by (14) in § 3. Let us condense these values by the following substitutions:

$$
\begin{array}{ll}
A = \sigma(b-c) & U_a = \sigma(u-a) \\
B = \sigma(c-a) & U_b = \sigma(u-b) \\
C = \sigma(a-b) & U_c = \sigma(u-c) \\
A_1 = \sigma_1(b-c) & U_{1a} = \sigma_1(u-a) \\
\text{etc.} & \text{etc.}
\end{array}
$$

Then

$$
\begin{aligned}
([illegible])\quad (\mathfrak{C}X) \equiv \big(& l_{20}^2 A_1 B C_1 U_{1a} U_{b} U_{1c} + l_{20}^2 A B_1 C_1 U_a U_{1b} U_{1c} \\
& + l_{30}^2 A_1 B_1 C\, U_{1a} U_{1b} U_c \big) X_0 \\
+ \big(& l_{01}^2 ABC\, \overline{l_1 - l_2}\, \overline{l_1 - l_3}\, U_a U_b U_c + l_{21}^2 A_1 B C_1 U_{1a} U_b U_{1c} \\
& + l_{31}^2 A_1 B_1 C\, U_{1a} U_{1b} U_c \big) X_1 + \quad + \quad = 0.
\end{aligned}
$$

This is a cubic in σ and so the tangent planes at the vertices envelop as a double-curve, this cubic.

If we call this envelope $(\xi x) = 0$ then the pole of ξ as to the quadric $(x^2_\gamma) = 0$, which in planes is $(\xi^2 \gamma) = 0$, is found at once by taking as the coordinates of this polar point, x, (of ξ) the coefficients of ξ' in $(\xi\xi'\gamma) = 0$ or $x_i = \xi_i \gamma_i$. If x is the vertex of the quadric, say v, then

$v_i = \xi_i \gamma_i$ represents the double point curve of Ω that is the intersection of the perpendicular pairs of generators. This is seen at once to be a sextic curve, and its equation is

(4) $$v_0 = c_0 y_0 = (l_{10}^2 y_1 + l_{20}^2 y_2 + l_{30}^2 y_3) y_0$$
$$= \Big[l_{10}^2 \frac{A^2 B C}{A_1^2 B_1 C_1} \frac{V_{1b} V_{1c}}{V_b V_c} + l_{20}^2 \frac{A B^2 C}{A_1 B_1^2 C_1} \frac{V_{1a} V_{1c}}{V_a V_c}$$
$$+ l_{30}^2 \frac{A B C^2}{A_1 B_1 C_1^2} \frac{V_{1a} V_{1b}}{V_a V_b} \Big] (l_1 - l_2)(l_1 - l_3)$$

$$v_1 = c_1 y_1 = l_{10}^2 y_0 y_1 + l_{12}^2 y_1 y_2 + l_{13}^2 y_1 y_3$$
$$= (l_1 - l_2)(l_1 - l_3)\, l_{10}^2 \frac{A^2 B C}{A_1^2 B_1 C_1} \frac{V_{1b} V_{1c}}{V_b V_c}$$

$$+ l_{12}^2 \frac{AB}{A_1B_1} \frac{V_{1a}V_{1b}V_1^2c}{V_aV_bV^2_c} + l_{13}^2 \frac{AC}{A_1C_1} \frac{V_1^2{}_aV_{1b}V_{1c}}{V^2_aV_bV_c}$$

and similar expressions for V_2 and V_3.

Multiplying by $A_1B_1C_1V_a^2V_b^2V_c^2$ we have:

(20) $V_0 = (l_1-l_2)(l_1-l_3)\left[A_{11}l_{10}^2 A_1BC\,V_a^2V_bV_cV_{1b}V_{1c}\right.$
$+ A_{22}l_{20}^2 AB_1C\,V_aV_b^2V_cV_{1a}V_{1c} + \left.A_{33}l_{30}^2 ABC_1V_aV_bV_c^2V_{1a}V_{1b}\right]$

$V_1 = (l_1-l_2)(l_1-l_3)\,A_{11}l_{10}^2 A_1BC\,V_a^2V_bV_cV_{1b}V_{1c}$
$+ l_{12}^2 ABC_1\,V_aV_bV_{1a}V_{1b}V_{1c}^2 + l_{13}^2 AB_1C\,V_bV_cV_{1a}^2V_{1b}V_{1c}$

and similar expressions for V_2 and V_3.

This sextic is only a part of the double-point curve, for the surface being represented by an equation of the ninth degree in an elliptic parameter, any plane section would be an elliptic 9-ic. and as such would necessarily have 27 double points. Therefore, <u>the complete double-point curve would be of degree 27</u> and the <u>double-plane curve being of degree</u>

three less, would be of degree 24. Then the complete double-point curve consists of a sextic, representing the intersections of those generators that meet in perpendicular pairs, and the rest of the intersections of the generators forming a curve of degree 21. There are 7 double points on every generator of Ω.

An interesting property of the planes on the perpendicular pairs of generators of Ω, i.e. those planes that envelop the cubic $(C\eta)=0$, is established as a result of the fact that these planes are the reciprocals of the points, y, on the cubic $\left(\frac{A^2}{y}\right)=0$, at infinity, with regard to the circumsphere of T', — a relation before mentioned. The points of any plane cubic reciprocate, with regard to a sphere, into planes

on a cubic cone, that is of class 3.
As the plane is the plane at infinity, and the reciprocal of this plane is the centre of the sphere, we have this fact;- all the planes touching the double-plane curve (β), pass thru the centre of the sphere,-that is the circumsphere of T.

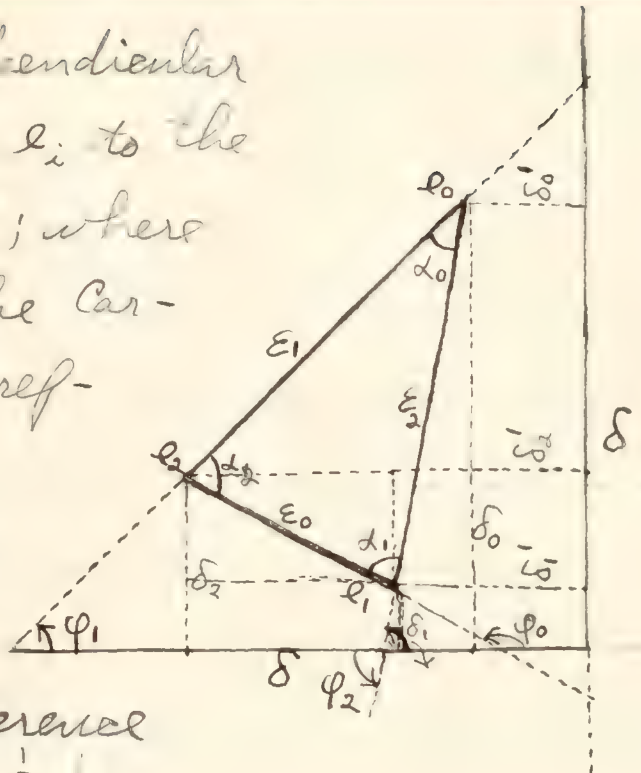
§.7. Relation between the Faces & Edges of a Tetrahedron

It is desirable to find the relations between the edges and the faces of T, in order to express the coefficients, in the foregoing expressions, entirely in terms of the areas of the faces. To find all such relations, we shall express the absolute in terms of the edges of T, and also in terms of the faces of T, & then equate coefficients.

First, let us find the equation of the absolute in the plane, in terms of the lengths of the edges of the fundamental triangle of reference.

Let e_0, e_1, e_2 be this triangle, and let the edges be $E_0\ E_1\ E_2$, where E_i is the side opposite the vertex e_i. Also let the angle at e_i be α_i, and δ_i and

δ_i' be the perpendicular distances from e_i to the lines δ and δ'; where δ and δ' are the Cartesian axes of reference.



Then the area of the triangle of reference is: $\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \delta_0 & \delta_1 & \delta_2 \\ \delta_0' & \delta_1' & \delta_2' \end{vmatrix}$

or, $2\Delta = -\delta_0(\delta_2' - \delta_1') + \delta_1(\delta_2' - \delta_0') - \delta_2(\delta_1' - \delta_0')$

Now let φ_i be the angle included between δ and ε_i where δ is to be taken as the initial line and the angle taken in a positive direction. This gives:

(21) $2\Delta = \delta_0 \varepsilon_0 \cos\varphi_0 + \delta_1 \varepsilon_1 \cos\varphi_1 - \delta_2 \varepsilon_2 \cos\varphi_2$

also $\begin{vmatrix} 1 & 1 & 1 \\ \delta_0 & \delta_1 & \delta_2 \\ \delta_0 & \delta_1 & \delta_2 \end{vmatrix} = 0$

or, $\delta_0(\delta_2 - \delta_1) + \delta_1(\delta_0 - \delta_2) - \delta_2(\delta_0 - \delta_1) = 0$

(22) Or, $\delta_0 \varepsilon_0 \sin\varphi_0 + \delta_1 \varepsilon_1 \sin\varphi_1 - \delta_2 \varepsilon_2 \sin\varphi_2 = 0$

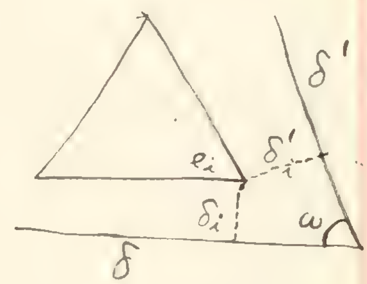
Squaring (21) and (22) and adding we have;

$$4\Delta^2 = \delta_0^2\varepsilon_0^2 + \delta_1^2\varepsilon_1^2 + \delta_2^2\varepsilon_2^2 - 2\delta_1\delta_2\varepsilon_1\varepsilon_2 \cos(\varphi_1 - \varphi_2) - 2\delta_2\delta_0\varepsilon_2\varepsilon_0\cos(\varphi_2 - \varphi_0) + 2\delta_0\delta_1\varepsilon_0\varepsilon_1\cos(\varphi_0 - \varphi_1)$$

$$\varphi_2 - \varphi_1 = \alpha_0, \quad \varphi_0 - \varphi_1 = 180^\circ - \alpha_2, \quad \varphi_0 - \varphi_2 = \alpha_1$$

Therefore $4\Delta^2 = \delta_0^2\varepsilon_0^2 + \delta_1^2\varepsilon_1^2 + \delta_2^2\varepsilon_2^2 - 2\delta_1\delta_2\varepsilon_1\varepsilon_2 \cos\alpha_0 - 2\delta_2\delta_0\varepsilon_2\varepsilon_0\cos\alpha_1 - 2\delta_0\delta_1\varepsilon_0\varepsilon_1\cos\alpha_2$

If δ and δ' are not perpendicular but form an angle, ω, with each other, the above equation becomes:



$$4\Delta^2\cos\omega = \delta_0\delta_0'\varepsilon_0^2 + \delta_1\delta_1'\varepsilon_1^2 + \delta_2\delta_2'\varepsilon_2^2 - \varepsilon_1\varepsilon_2(\delta_1\delta_2' + \delta_1'\delta_2)\cos\alpha_0 - \varepsilon_2\varepsilon_0(\delta_2\delta_0' + \delta_2'\delta_0)\cos\alpha_1 - \varepsilon_0\varepsilon_1(\delta_0\delta_1' + \delta_0'\delta_1)\cos\alpha_2$$

The δ's are proportional to the coordinates, ξ and ξ' of the lines δ and δ'. Substituting ξ_i and ξ_i' for δ_i and δ_i' and factoring we have:

$$4\Delta^2\cos\omega = (\xi_0\varepsilon_0 e^{i\alpha_0} + \xi_1\varepsilon_1 e^{i\alpha_1} + \xi_2\varepsilon_2 e^{i\alpha_2})(\xi_0'$$

$z_0 e^{-i\alpha_0} + \xi_1' z_1 e^{-i\alpha_1} + \xi_2' z_2 e^{-i\alpha_2})$ or

$$4\Delta^2 = \frac{1}{\cos\omega}\left(\xi z e^{i\alpha}\right)\left(\xi' z e^{-i\alpha}\right)$$

and $4\Delta^2 = \infty$ for $\omega = 90^0$

(23) Therefore $(\xi z e^{i\alpha})(\xi' z e^{-i\alpha}) = 0$ is the equation of the absolute.

This is also verified to be the absolute, thus:

$$z_0^2 = z_1^2 + z_2^2 - 2 z_1 z_2 \cos\alpha_0$$

$$\Delta = \frac{z_1 z_2 \sin\alpha_0}{2}$$

and cotangent $\alpha_0 = \dfrac{z_1^2 + z_2^2 - z_0^2}{4\Delta}$ = say c_0

with similar expressions for c_1 and c_2. also $c_1 + c_2 = \dfrac{z_0^2}{2\Delta}$, $c_2 + c_0 = \dfrac{z_1^2}{2\Delta}$, $c_0 + c_1 = \dfrac{z_2^2}{2\Delta}$

Substituting these values for the z_i's in (23) we get:

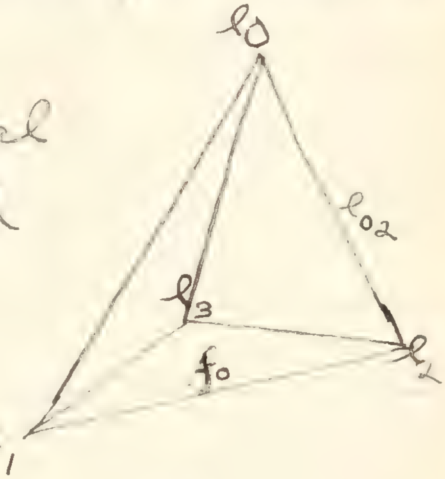
$$(c_1+c_2)\xi_0^2 + (c_2+c_0)\xi_1^2 + (c_0+c_1)\xi_2^2$$
$$-2c_0\xi_1\xi_2 - 2c_1\xi_2\xi_0 - 2c_2\xi_0\xi_1 = 0.$$

This is easily seen to represent a pair of points on the line at infinity, i.e. is the absolute in the plane.

Equation (23) can be written in

the following form:

(24) $\sum^{6} \varepsilon_i \varepsilon_j \xi_i \xi_j \cos(ij) = 0$ where i and j can equal 0, 1, 2, . , and $\cos(ij)$ stands for cosine of the angle between ε_i and ε_j.



There are two natural extensions of (24) which equally well would represent the absolute in space. If we represent the four vertices of the tetrahedron of reference by e_0, e_1, e_2, and e_3, and the face opposite e_i by f_i, (i.e. the area of this face is f_i) the space analogue of (24) would be:

(25) $\sum^{10} f_i f_j \xi_i \xi_j \cos(ij) = 0$. Here ξ_i and ξ_j are line coordinates and the angle (ij) is the dihedral angle between the faces f_i and f_j. (i and $j = 0, 1, \ldots, 3$). Let A_{ij} represent $f_i f_j \cos(ij)$, and equation

(25) takes the more simple form:

(26) $(A\xi)^2 \equiv A_{00}\xi_0^2 + A_{11}\xi_1^2 + A_{22}\xi_2^2 + A_{33}\xi_3^2$
$+ A_{01}\xi_0\xi_1 + A_{02}\xi_0\xi_2 + \quad + \quad + \quad + \quad = 0.$

As a second extension of (27) let us write a similar equation of such kind that it will represent the absolute in each plane that is a side of T. A_{ij} as defined above is, for any face, the square of the area of that face, for, $A_{ii} = f_i f_i \cos(ii) = f_i^2$, and so we will speak of A_i as the area of the face f_i. Also let l_{ij} be the edge of T that passes thru the vertices i and j. Then in each of the faces, the absolute is:

A_0: $\sum^6 \varepsilon_i \varepsilon_j \xi_i \xi_j \cos(ij) = 0$, ($i$ and $j = 1, 2, 3$)
A_1: $\sum^6$ " $= 0$ (" $= 0, 2, 3$)
A_2: $\sum^6$ " $= 0$ (" $= 0, 1, 3$)
A_3: $\sum^6$ " $= 0$ (" $= 0, 1, 2$)

The equation that will include

these four is:

(27) $\sum^{21} e_i \varepsilon_j \xi_i \xi_j \cos(ij) = 0$. ($i$ and $j = 0, 1, 2, 3$.)

where ε_i is the edge of T opposite the vertex, e_i, of each face that is on e_i. (The slight difficulty in defining the edges ε_i will be removed in the next form in which the above equation is written by the substitution of e_{ij} for ε_K.)

We shall next express equations (26) and (27) in line coordinates, π_{ij}, where by π_{ij} we mean the minors of the matrix
$$\begin{Vmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{Vmatrix}.$$

First let us determine these coordinates for a line in each of the reference planes. The line π is taken as an axis and ξ and η are planes on this axis, and if π is in a reference plane, then the planes ξ and η will

being ~~ines~~ on this ~~p~~line, ~~i.e.~~ the coordinates ξ_i and η_i will be line coordinates in this plane, where η is one of the reference planes.

In the plane

$A_0 \ (\eta_0 = 0)$ $\quad$ $A_1 \ (\eta_1 = 0)$

$$\pi = \left\| \begin{matrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ 1 & 0 & 0 & 0 \end{matrix} \right\| \qquad \pi = \left\| \begin{matrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 & 0 \end{matrix} \right\| \qquad \text{etc.}$$

or from these matrices, and the quadratic identity between the plane and point coordinates of a line, we have:

In plane A_0 ; in A_1 ;

$\xi_1 = \pi_{10} = \rho p_{32}$ $\quad$ $\xi_0 = \pi_{01} = \rho p_{23}$

$\xi_2 = \pi_{20} = \rho p_{13}$ $\quad$ $\xi_2 = \pi_{21} = \rho p_{30}$

$\xi_3 = \pi_{30} = \rho p_{21}$ $\quad$ $\xi_3 = \pi_{31} = \rho p_{02}$

in A_2 ; and in A_3

$\xi_0 = \pi_{02} = \rho p_{31}$ $\quad$ $\xi_0 = \pi_{03} = \rho p_{12}$

$\xi_1 = \pi_{12} = \rho p_{03}$ $\quad$ $\xi_1 = \pi_{13} = \rho p_{20}$

$\xi_3 = \pi_{32} = \rho p_{10}$ $\quad$ $\xi_2 = \pi_{23} = \rho p_{01}$

Substituting these values in (1)

and also substituting l_{ij} for $\mathcal{E}_K$, we

yet:

(26) $$\rho^2 [l_{12} l_{02} p_{12} p_{20} \cos(l_{12} l_{02})$$
$$+ l_{02} l_{01} p_{20} p_{01} \cos(l_{02} l_{01}) + l_{12} l_{10} p_{01} p_{12} \cos(l_{12} l_{10})$$
$$+ l_{02} l_{30} p_{30} p_{02} \cos(l_{02} l_{30}) + l_{30} l_{23} p_{23} p_{30} \cos(l_{30} l_{23})$$
$$+ l_{02} l_{23} p_{02} p_{23} \cos(l_{02} l_{23}) + l_{31} l_{10} p_{31} p_{10} \cos(l_{31} l_{10})$$
$$+ l_{03} l_{31} p_{03} p_{31} \cos(l_{03} l_{31}) + l_{10} l_{03} p_{10} p_{03} \cos(l_{10} l_{03})$$
$$+ l_{32} l_{21} p_{32} p_{21} \cos(l_{32} l_{21}) + l_{13} l_{32} p_{13} p_{32} \cos(l_{13} l_{32})$$
$$+ l_{21} l_{13} p_{21} p_{13} \cos(l_{21} l_{13}) + l_{12} l_{03} p_{12} p_{03} \cos(l_{12} l_{03})$$
$$+ l_{20} l_{13} p_{20} p_{13} \cos(l_{20} l_{13}) + l_{01} l_{23} p_{01} p_{23} \cos(l_{01} l_{23})$$
$$- l_{12}^2 p_{12}^2 - l_{20}^2 p_{20}^2 - l_{01}^2 p_{01}^2 - l_{03}^2 p_{03}^2 - l_{13}^2 p_{13}^2 - l_{23}^2 p_{23}^2] = 0.$$

Now let us express equation (26) in p coordinates. The line equation is, in determinant form,

$$\begin{vmatrix} A_{00} & A_{01} & A_{02} & A_{03} & x_0 & y_0 \\ A_{01} & A_{11} & A_{12} & A_{13} & x_1 & y_1 \\ A_{02} & A_{12} & A_{22} & A_{23} & x_2 & y_2 \\ A_{03} & A_{13} & A_{23} & A_{33} & x_3 & y_3 \\ x_0 & x_1 & x_2 & x_3 & 0 & 0 \\ y_0 & y_1 & y_2 & y_3 & 0 & 0 \end{vmatrix} = 0.$$

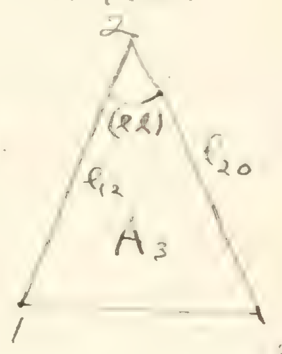
This is to be developed in terms

of the minors of $\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{matrix} \right\|$

and gives an expression consisting of:

$$\left.\begin{array}{l} 12 \text{ terms like } +(A_{03}A_{13} - A_{01}A_{33})p_{12}p_{20} \\ \qquad\qquad + (A_{13}A_{23} - A_{12}A_{33})p_{20}p_{01} \\ \text{plus 3 terms like } 2(A_{01}A_{23} - A_{02}A_{13})p_{12}p_{03} \\ \qquad\qquad + 2(A_{03}A_{12} - A_{01}A_{23})p_{20}p_{13} \\ \text{plus 6 terms like } (A_{03}^2 - A_{00}A_{33})p_{12}^2 \\ \qquad\qquad\qquad = 0. \end{array}\right. \tag{29}$$

Now $l_{02}^2 l_{12}^2 \sin^2(l_{02}l_{12}) = l_{02}^2 l_{12}^2 [1 - \cos^2(l_{02}l_{12})] = 4\Delta^2 = 4A_{33}$



$\therefore l_{02}l_{12}\cos(l_{02}l_{12}) = \sqrt{l_{02}^2 l_{12}^2 - 4A_{33}}$

and similarly expressions can be derived for the quantities, $l_{ij}l_{ik}\cos(l_{ij}l_{ik})$.

These expressions substituted in (28) give

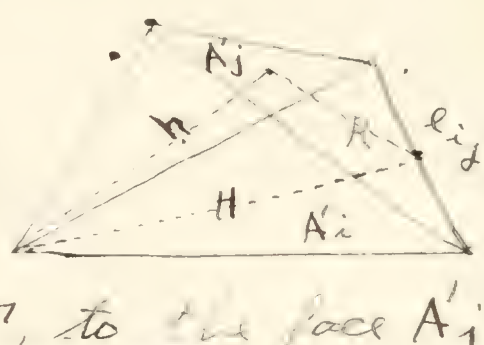
$$\begin{array}{l} [12 \text{ terms like } \sqrt{l_{12}^2 l_{20}^2 - 4A_{33}}\; p_{12}p_{20} \\ \qquad\qquad + \sqrt{l_{20}^2 l_{01}^2 - AA_{33}}\; p_{20}p_{01} \end{array} \tag{30}$$

plus 3 terms like $l_{12} l_{03} \cos(l_{12} l_{03}) p_{12} p_{03}$
plus 6 terms like $-l_{12}^2 p_{12}^2 \Big] = 0$

Now these equations, (29) and (30), represent the absolute and are both written in terms of the same coordinates and consequently their coefficients, term for term, must be the same. The equating of these 21 coefficients will give all the relations existing between the faces and edges of T'.

We shall next determine the value of ρ.

Take the two faces, A_i' and A_j', meeting in the edge l_{ij}. Let



h be the altitude of T', to the face A_j'; and H be the altitude, from the same vertex, of the face A_i'. Let V be the volume of T', and R the sin of

the foot of h and the foot of H.

Then $3V = h A'_j$ and $2A'_i = l_{ij} H$

The triangle hRH is a right triangle and moreover the dihedral angle at l_{ij} is measured by the plane angle between R and H. Consequently

$$p = H \sin(l_{ij})$$

These last three equations give

$$\frac{3V}{2} l_{ij} = A_i A_j \sin(l_{ij})$$

Equating the last coefficients of (29) and (30) we get:

$$\rho^2 l_{ij}^2 = A_{ii} A_{jj} - A_{ij}^2$$

As $A_{ij} = A_i A_j \cos(l_{ij})$ we have:

$$\rho^2 l_{ij}^2 = A_{ii} A_{jj} [1 - \cos^2(l_{ij})] =$$

$$A_{ii} A_{jj} \sin^2(l_{ij})$$

$\therefore \rho l_{ij} = A_i A_j \sin(l_{ij})$

and $\rho l_{ij} = \frac{3V}{2} l_{ij}$ or $\rho = \frac{3V}{2}$

Equating now the 21 coefficients above and substituting this value of ρ we have the following 21

relations between the faces and edges of Π.

$$\left(\frac{3V}{2}\right)^2 l_{12}^2 = A_{00}A_{33} - A_{03}^2$$
$$\left(\frac{3V}{2}\right)^2 l_{20}^2 = A_{11}A_{33} - A_{13}^2$$

and four similar relations.

$$\left(\frac{3V}{2}\right)^4 l_{12}^2 l_{20}^2 = (A_{03}A_{13} - A_{01}A_{33})^2 + 4A_{33}\left(\frac{3V}{2}\right)^4$$
$$\left(\frac{3V}{2}\right)^4 l_{20}^2 l_{01}^2 = (A_{13}A_{23} - A_{12}A_{33})^2 + 4A_{33}\left(\frac{3V}{2}\right)^4$$
$$\left(\frac{3V}{2}\right)^4 l_{01}^2 l_{12}^2 = (A_{03}A_{23} - A_{02}A_{33})^2 + 4A_{33}\left(\frac{3V}{2}\right)^4$$

and nine similar relations

$$\left(\frac{3V}{2}\right)^2 l_{12}l_{03}\cos(l_{12}l_{03}) = 2(A_{01}A_{23} - A_{02}A_{13})$$
$$\left(\frac{3V}{2}\right)^2 l_{02}l_{13}\cos(l_{02}l_{13}) = 2(A_{03}A_{12} - A_{01}A_{23})$$
$$\left(\frac{3V}{2}\right)^2 l_{01}l_{23}\cos(l_{01}l_{23}) = 2(A_{02}A_{13} - A_{03}A_{12})$$

The last 3 equations may be written without the cosine function, by use of the following relations given by Ferrers.*

$$2\, l_{12}l_{30}\cos(l_{12}l_{30}) = (l_{01}^2 + l_{23}^2) - (l_{02}^2 + l_{13}^2)$$

(and 2 similar ones.)

Making this substitution, the above 3 become

$$\left(\frac{3V}{2}\right)^2\left[(l_{01}^2 + l_{23}^2) - (l_{02}^2 + l_{13}^2)\right] = 4(A_{01}A_{23} - A_{13}A_{20})$$

* Quarterly Journal Vol. 3, p 145.

$$\left(\frac{3V}{2}\right)^2[(l_{12}^2+l_{03}^2)-(l_{01}^2+l_{23}^2) = 4(A_{03}A_{12}-A_{01}A_{23})$$

$$\left(\frac{3V}{2}\right)^2[(l_{02}^2+l_{13}^2)-(l_{12}^2+l_{03}^2) = 4(A_{02}A_{13}-A_{03}A_{12})$$

These relations enable us to express all the constants in the equations of this paper in terms of either the areas of the faces, or the lengths of the edges of the tetrahedron of reference.

§ 8. Surfaces connected with Ω

The vertices of Γ are [illegible] [illegible] [illegible] [illegible] that is [illegible] [illegible] [illegible] [illegible] [illegible] the [illegible] of the principal [illegible] of the [illegible] with which we are concerned. This is seen to be true in the following way.

The vertices of Γ, as determined in § 2, are (a,b,c), (d,e,f), (r,s,t), $(\frac{B}{C}, -\frac{A}{C}, \ldots)$. Their projections on the above plane, which we take as the xy plane, are (a,b), (d,e), (r,s), $(\frac{B}{C}, -\frac{A}{C})$. The condition for these four points to be on a circle is the vanishing of the determinant,

$$\Delta \equiv \begin{vmatrix} a^2+b^2 & a & b & 1 \\ d^2+e^2 & d & e & 1 \\ r^2+s^2 & r & s & 1 \\ \left(\frac{A}{C}\right)^2+\left(\frac{B}{C}\right)^2 & \frac{B}{C} & -\frac{A}{C} & 1 \end{vmatrix}$$

which expanded is:

$$\Delta \equiv (a^2+b^2)\left[(ds-er)C+(d-r)A+(e-s)B\right]$$
$$+(d^2+e^2)\left[(br-as)C+(r-a)A+(s-b)B\right]$$
$$+(r^2+s^2)\left[(ae-bd)C+(a-d)A+(b-e)B\right]$$
$$-(A^2+B^2)$$

Now $A = b(f-t)+e(t-c)+s(c-f)$

$B = a(f-t)+d(t-c)+r(c-f)$

$C = ae+br+ds-as-bd-er$

and $(f-t) = bd+ae-br-as$

$(t-c) = er+ds-bd-ae$

$(c-f) = br+as-er-ds$

$$\therefore \Delta \equiv (a^2+b^2)(e^2r^2-d^2s^2)+(d^2+e^2)(a^2s^2-b^2r^2)$$
$$+(r^2+s^2)(b^2d^2-a^2e^2) = 0$$

$\therefore$ These 4 points are on a circle in the plane, $z=0$, and consequently the 4 vertices of T are on a circular cylinder perpendicular to the plane of the principal generators, i.e., the plane tangent to the paraboloid at the vertex.

The vertices of T' and T'' are

on cylinders of the same radius as T and symmetrically placed with regard to the axes. T and T' are symmetrically placed as to the x-axis, T and T'', as to the y-axis and T' and T'' are consequently each the reflection of the other in the z-axis. T' and T'' are also inscribed in the above cylinders as they only differ from T in the change of sign of the y and x coordinates, respectively, and this doesn't affect the vanishing of Δ.

If we had started with T' as the first tetrahedron and reflected it in the x-axis we should have had T, and the reflection of T' in the z-axis would have given us a fourth tetrahedron T'''. T''' would be the reflection of T'' in the x-axis and of T

in the z-axis. Thus from any given tetrahedron as a tetrahedron of reference, which is self-polar with regard to an orthic paraboloid, we get two others, T' and T'', each inscribed to T and a fourth, T''', which is the reflection of T in the z-axis, T' in the y-axis, and T'' in the x-axis. Each of the four tetrahedra are inscribed in a cylinder whose generators are perpendicular to the plane that is tangent to the quadric at its vertex. Moreover the axes of these 4 cylinders are on a fifth cylinder whose axis is the z axis, and these 5 cylinders meet the plane, W, in the cubic curve, $(A^2_y) = 0_x$ *

That a cylinder should contain 4 points is, in general, only 3×4−4 or 8 conditions. As the number of con-

* Beltrami has proven that the axes of the cylinders on 4 points, intersect the plane at infinity, in this cubic curve as the edges of T are obviously axes

ditions that are necessary to determine a cylinder is 9, the cylinder on 8 points has one degree of freedom and consequently there are ∞^1 such cylinders and their axes must lie in some ruled surface.

J. B. Eck has shown this surface to be also of the same degree as Ω, i.e. a ruled surface of the ninth degree, and has discussed this surface in some detail.*

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* "Über die Verteilung der Axen der Rotationsflächen 2 Grades, welche durch gegebene Punkte gehen." Königlichen Akademie zu Münster, 1890. (Dissertation.)

This ruled surface is definitely connected with Ω. They are both of the same degree and there is a direct correlation between the generators of the first and those planes of the second that contain perpendicular generators. For each such plane of Ω there is one generator of the first surface that is perpendicular to it.

The axes of the paraboloids of this paper also form a ruled surface, whose degree can not be more than nine, for the generators of it all pass thru the sextic, (11), and the cubic curve, $\left|\frac{A^2}{4}\right| = 0$, at infinity. The axes of the cylinders on the tetrahedra T', T'', and T''', are on ruled surfaces and these last four surfaces all pass thru the cubic curve $\left|\frac{A^2}{4}\right| =$

§9. The Quadratic Transformation.

The equations (31) represent a transformation, that is necessarily quadratic. Let us write these equations as;

(31) $\quad x_i = c_i y_i = \alpha_{ij} y_{ij} + \alpha_{ik} y_{ik} + \alpha_{il} y_{il}$

where $i \neq j \neq k \neq l$ and $i, j, k, l = 0, 1, 2, 3$.

Let us consider the correspondence this establishes between two spaces, S_x and S_y. Substituting these expressions in $(\xi x) = 0$, we have the correspondence in the form:

(32) $(\xi c y) = 0$. or $\sum_0^3 \xi_i [\alpha_{ij} y_{ij} + \alpha_{ik} y_{ik} + \alpha_{il} y_{il}] = 0$.

This gives for a plane ξ in S_x, a quadric in S_y; for a line in S_x, a quartic curve in S_y; and consequently for a point in S_x, 8 pts. in S_y. All planes of S_x give a 3-fold system of quadrics in S_y.

a plane in S_y meets a quartic curve in 4 points and consequently, a line in S_x meets the correspondent of a plane in S_y, in 4 points. Therefore, this correspondent is a quartic surface and can be shown to be a Steiner's Quartic Surface. To a line in S_y corresponds a conic in S_x. A plane η meets the 3-fold system of quadrics in a 5-fold system of conics which map the plane η onto its corresponding Steiner's Quartic Surface.

The Jacobian of the system of quadrics is a quartic surface and contains the ten lines that are the intersections of the planes that represent the ten degenerate quadrics of the system.

The Jacobian of the system of

quadrics, is:

$$\begin{vmatrix} \alpha_{01}y_1+\alpha_{02}y_2+\alpha_{03}y_3, & \alpha_{01}y_0, & \alpha_{02}y_0 & \alpha_{03}y_0 \\ \alpha_{01}y_1 & \alpha_{01}y_0+\alpha_{12}y_2+\alpha_{13}y_3 & \alpha_{12}y_1 & \alpha_{13}y_1 \\ \alpha_{02}y_2 & \alpha_{12}y_2 & \alpha_{02}y_0+\alpha_{12}y_1+\alpha_{23}y_3, & \alpha_{23}y_2 \\ \alpha_{03}y_3 & \alpha_{13}y_3 & \alpha_{23}y_3, & \alpha_{03}y_0+\alpha_{13}y_1+\alpha_{23}y_2 \end{vmatrix}=0$$

or

$$(3.3)\quad \mathcal{J} \equiv \alpha_{01}\alpha_{02}\alpha_{12}\, y_0y_1y_2(\alpha_{03}y_0+\alpha_{13}y_1+\alpha_{23}y_2)$$
$$+\alpha_{01}\alpha_{03}\alpha_{13}\, y_0y_1y_3(\alpha_{02}y_0+\alpha_{12}y_1+\alpha_{23}y_3)$$
$$+\alpha_{02}\alpha_{03}\alpha_{23}\, y_0y_2y_3(\alpha_{01}y_0+\alpha_{12}y_2+\alpha_{13}y_3)$$
$$+\alpha_{12}\alpha_{13}\alpha_{23}\, y_1y_2y_3(\alpha_{01}y_0+\alpha_{02}y_2+\alpha_{03}y_3)=0$$

or, $\mathcal{J} \equiv \sum^{4} \alpha_{ij}\alpha_{ik}\alpha_{jk}\, y_iy_jy_k(\alpha_{il}y_i+\alpha_{jl}y_j+\alpha_{kl}y_k)=0$

Since to a line in S_x, corresponds an elliptic quartic in S_y, the correspondent of $\mathcal{J}$ will meet the line in the same number of points as a quartic curve meets a surface of order four, or sixteen points. Then _the correspondent of $\mathcal{J}$ is a surface of order sixteen_.

The equation of the correspondent of J will be given in planes, as the discriminant of (32), for a plane ξ will touch this correspondent when ξ touches the quadrics, i.e. when the discriminant vanishes. This gives

$$\begin{vmatrix} 0 & \alpha_{01} & \alpha_{02} & \alpha_{03} \\ \alpha_{01} & 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{02} & \alpha_{12} & 0 & \alpha_{23} \\ \alpha_{03} & \alpha_{13} & \alpha_{23} & 0 \end{vmatrix} = 0$$

where α_{01} represents $\alpha_{01}(\xi_0+\xi_1)$, etc.

or $\alpha_{01}^2\alpha_{23}^2 + \alpha_{02}^2\alpha_{13}^2 + \alpha_{03}^2\alpha_{12}^2$

$$-2(\alpha_{01}\alpha_{02}\alpha_{13}\alpha_{23} + \alpha_{02}\alpha_{03}\alpha_{12}\alpha_{13} + \alpha_{01}\alpha_{03}\alpha_{12}\alpha_{23}) = 0,$$

or $(\alpha_{01}\alpha_{23} + \alpha_{02}\alpha_{13} - \alpha_{03}\alpha_{12})^2 = 4\alpha_{01}\alpha_{02}\alpha_{13}\alpha_{23}$

This is at once seen to be

$$(34)\quad Q:\ \sqrt{\alpha_{01}\alpha_{23}(\xi_0+\xi_1)(\xi_2+\xi_3)} + \sqrt{\alpha_{02}\alpha_{13}(\xi_0+\xi_2)(\xi_1+\xi_3)} + \sqrt{\alpha_{03}\alpha_{12}(\xi_0+\xi_3)(\xi_1+\xi_2)} = 0.$$

This is seen to be of the 4th degree in ξ and therefore the correspondent

of $\mathcal{A}$ is a surface of class 7.

As the reflection of the plane, $\xi_0+\xi_1=0$, in the centroid of T whose coordinates are $(1,1,1,1)$ is $\xi_2+\xi_3$, and similarly for $\xi_0+\xi_2$ and $\xi_0+\xi_3$ which reflect into $\xi_1+\xi_3$ and $\xi_1+\xi_2$, we see that,–

The surface $\mathcal{L}$ is symmetrical with regard to the centroid of the tetrahedron of reference.

Let us now consider the correspondence between a plane ξ and its corresponding Steiner's quartic surface. In particular, let this plane be the plane, W at infinity. The equation of this surface is found at once by forming the Hessian of $(\xi c y)$, and after this equation is changed over into the point equation, make the surface touch the plane W. This gives in determinant form;

$$\begin{vmatrix} 0 & \alpha_{01} & \alpha_{02} & \alpha_{03} & 1 \\ \alpha_{01} & 0 & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{02} & \alpha_{12} & 0 & \alpha_{23} & 1 \\ \alpha_{03} & \alpha_{13} & \alpha_{23} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

where as before α_{01} represents $\alpha_{01}(\xi_0+\xi_1)$ &c.

This leveled out is

(35) $R \equiv (\alpha_{12}+\alpha_{13}-\alpha_{23})(\alpha_{01}\alpha_{23}+\alpha_{02}\alpha_{03})$
$+ (\alpha_{12}-\alpha_{13}+\alpha_{23})(\alpha_{02}\alpha_{13}+\alpha_{01}\alpha_{03})$

$$+(-\alpha_{12}+\alpha_{13}+\alpha_{23})(\alpha_{03}\alpha_{12}+\alpha_{01}\alpha_{02})$$
$$=\alpha_{01}^2\alpha_{23}+\alpha_{02}^2\alpha_{13}+\alpha_{03}^2\alpha_{12}+\alpha_{12}\alpha_{13}\alpha_{33}.$$

The cubic C_3, i.e. the intersection of W and Ω, maps into a sextic curve on R, the sextic (19), which is a part of the double-point curve on Ω. The quadric corresponding to the plane, W, whose coordinates are 1, 1, 1, 1, is seen from $(\xi c \gamma)=0$, to be $(c\gamma)=0$, which is the equation of the sphere circumscribed to T. So this quadric is one of the system, and the plane, W, meets it in the absolute, which, consequently, is one of the conics of the 3-fold system in W. Thus to the plane W, in the space S_x, corresponds the quadric $(c\gamma)=0$, in S_y, and to the plane W, in S_y, corresponds the quartic surface R in S_x. To the inter-

section of $(cy)=0$, and W, i.e. to the absolute, will correspond a rational quartic, the intersection of W and R. The absolute meets the cubic C_3 in 6 points and therefore this rational quartic in W will be on 6 points of the sextic (19). These two sets of 6 points are one and the same.

To the 4 lines, $y_i=0$, which are the intersections of T and W, will correspond a second set of 4 lines, whose equations are $c_i=0$, i.e. are

$$l_{01}^2 y_1 + l_{02}^2 y_2 + l_{03}^2 y_3 = 0.$$
$$l_{01}^2 y_0 + l_{12}^2 y_2 + l_{13}^2 y_3 = 0.$$
$$l_{02}^2 y_0 + l_{12}^2 + y_1 + l_{23}^2 y_3 = 0$$
$$l_{03}^2 y_0 + l_{13}^2 + y_1 + l_{23}^2 y_2 = 0$$

Hesse has shown that if T and T' are 2 tetrahedra, self-polar as to a quadric, their vertices are a set of 8 associated points.

The converse is equally true. i.e. for 8 associated points, which can be that of as 2 tetrahedra there is a definite quadric to which the tetrahedra are self-polar.

Suppose $(\alpha x)^2$, $(\beta x)^2$, $(\gamma x)^2$ to be 3 quadrics on the 8 points of an associated set, and that 4 of these points form the vertices of the tetrahedron of reference and let us name the other four a, b, c, d. Let (λx^2) be the quadric to which both tetrahedra are self-polar. The polar of x' as to $(\lambda x^2) = 0$, is $(\lambda x x') = 0$ and of $\underline{a}$ is $(\lambda a x) = 0$. If this plane is to be on b, then $(\lambda a b) = 0$. Similarly $(\lambda a c) = 0$, and $(\lambda b c) = 0$. These

3 conditions on $a, b,$ and c will make the tetrahedron a, b, c, d self-polar as to $(\lambda x^2)=0$, for then $\underline{d}$ is determined.

Eliminating λ between the last 4 equations we have the quadric $(\lambda x^2)=0$ in this form:

$$(36)\quad \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ a_0b_0 & a_1b_1 & a_2b_2 & a_3b_3 \\ b_0c_0 & b_1c_1 & b_2c_2 & b_3c_3 \\ c_0a_0 & c_1a_1 & c_2a_2 & c_3a_3 \end{vmatrix} = 0$$

If $(\alpha x)^2, (\beta x)^2, (\gamma x)^2$ respectively are on a, b, c, then

$$(37)\quad \begin{cases} \alpha_{01}a_0a_1+\alpha_{02}a_0a_2+\alpha_{03}a_0a_3+\alpha_{12}a_1a_2+\alpha_{13}a_1a_3+\alpha_{23}a_2a_3=0 \\ \beta_{01}b_0b_1+\beta_{02}b_0b_2+ \quad + \quad + \quad + \quad =0 \\ \gamma_{01}c_0c_1+\gamma_{02}c_0c_2+ \quad + \quad + \quad + \quad =0 \end{cases}$$

From these 3 equations we have

$$(38)\quad \begin{vmatrix} \alpha_{ij} & \alpha_{ik} & \alpha_{il} \\ \beta_{ij} & \beta_{ik} & \beta_{il} \\ \gamma_{ij} & \gamma_{ik} & \gamma_{il} \end{vmatrix} = \rho \begin{vmatrix} a_ka_l, & a_la_j, & a_ja_k \\ b_kb_l & b_lb_j & b_jb_k \\ c_kc_l & c_lc_j & c_jc_k \end{vmatrix}$$

i.e. the $\alpha, \beta,$ and γ's expressed in terms

of the coordinates of the points a, b, c.
Now

(39) $$\begin{vmatrix} a_k a_l, & a_l a_j, & a_j a_k \\ b_k b_l, & b_l b_j, & b_j b_k \\ c_k c_l, & c_l c_j, & c_j c_k \end{vmatrix} = \begin{vmatrix} a_l b_l, & a_j b_j, & a_k b_k \\ b_l c_l, & b_j c_j, & b_k c_k \\ c_l a_l, & c_j a_j, & c_k a_k \end{vmatrix}$$

By means of these last two relations, the coefficients of x_i^2 in (36) can be expressed in terms of α, β and γ. Thus the coefficient of x_0^2 is:

$$\begin{vmatrix} a_1 b_1, & a_2 b_2, & a_3 b_3 \\ b_1 c_1, & b_2 c_2, & b_3 c_3 \\ c_1 a_1, & c_2 a_2, & c_3 a_3 \end{vmatrix} = \begin{vmatrix} a_1 a_3, & a_1 a_2, & a_2 a_3 \\ b_1 b_3 & b_1 b_2 & b_2 b_3 \\ c_1 c_3 & c_1 c_2 & c_2 c_3 \end{vmatrix} = \rho \begin{vmatrix} \alpha_{02}, & \alpha_{03}, & \alpha_{01} \\ \beta_{02}, & \beta_{03}, & \beta_{01} \\ \gamma_{02}, & \gamma_{03}, & \gamma_{01} \end{vmatrix}$$

(36) then becomes:

(40) $$x_0^2 |\alpha_{01} \beta_{02} \gamma_{03}| + x_1^2 |\alpha_{01} \beta_{13} \gamma_{12}| + x_2^2 |\alpha_{02} \beta_{12} \gamma_{23}| + x_3^2 |\alpha_{03} \beta_{23} \gamma_{13}| = 0$$

We shall now apply this to the particular case at hand.

In the special case of this section the 8 points, y, corresponding to a given point x, break up into 2 sets

of 4 points each, namely:- the one set that represents the fundamental tetrahedron, and which consequently is fixed and common to all sets of 8 points in S_y that correspond to points in S_x; the second set of 4 points, which is different for each set of 8.

Let the fixed set be $x_i = 0$. $(i = 0, 1, 2, 3)$ Then if y be one of the remaining set of 4 points, and y' any one of the remaining 3, there would be these relations, arising from $x_i = c_i y_i$

$$\frac{c_0 y_0}{c_0' y_0'} = \frac{c_1 y_1}{c_1' y_1'} = \frac{c_2 y_2}{c_2' y_2'} = \frac{c_3 y_3}{c_3' y_3'}$$

From these 3 equations, 3 quadratics arise:

$$(41)\begin{cases} c_0 c_1' y_0 y_1' = c_0' c_1 y_0' y_1 \\ c_0 c_2' y_0 y_2' = c_0' c_2 y_0' y_2 \\ c_0 c_3' y_0 y_3' = c_0' c_3 y_0' y_3 \end{cases}$$

which, if we call $\frac{e_0\gamma_0}{e_i\gamma_i} = A_i$, become

$$e_0'\gamma_0' = A_1 e_1'\gamma_1' = A_2 e_2'\gamma_2' = A_3 e_3'\gamma_3'.$$

Or, if we write γ_{01}' for $\gamma_0'\gamma_1'$,

$$(42)\begin{cases} l_{01}^2\gamma_{01}' + l_{02}^2\gamma_{02}' + l_{03}^2\gamma_{03}' = A_1(l_{01}^2\gamma_{01}' + l_{12}^2\gamma_{12}' + l_{13}^2\gamma_{13}') \\ \qquad\qquad '' \qquad\qquad\qquad = A_2(l_{02}^2\gamma_{02}' + l_{12}^2\gamma_{12}' + l_{23}^2\gamma_{23}') \\ \qquad\qquad '' \qquad\qquad\qquad = A_3(l_{03}^2\gamma_{03}' + l_{13}^2\gamma_{13}' + l_{23}^2\gamma_{23}') \end{cases}$$

These 3 quadrics all pass thru the vertices of T and thru the other set of 4 points, T'. To get the quadric with regard to which, T & T' are self-polar, we will only have to substitute in (40) the following values for α, β, and γ.

$\alpha_{01} = l_{01}^2(1-A_1)$ $\quad$ $\alpha_{02} = l_{02}^2$ $\quad$ $\alpha_{03} = l_{03}^2$

$\beta_{01} = l_{01}^2$ $\quad$ $\beta_{02} = l_{02}^2(1-A_2)$ $\quad$ $\beta_{03} = l_{03}^2$

$\gamma_{01} = l_{01}^2$ $\quad$ $\gamma_{02} = l_{02}^2$ $\quad$ $\gamma_{03} = l_{03}^2(1-A_3)$

$\alpha_{12} = -l_{12}^2 A_1$ $\quad$ $\alpha_{13} = -l_{13}^2 A_1$ $\quad$ $\alpha_{23} = 0$

$\beta_{12} = -l_{12}^2 A_2$ $\quad$ $\beta_{13} = 0$ $\quad$ $\beta_{23} = -l_{23}^2 A_2$

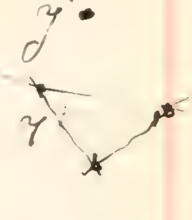
$\gamma_{12} = 0$ $\quad$ $\gamma_{13} = -l_{13}^2 A_3$ $\quad$ $\gamma_{23} = -l_{23}^2 A_3$

The result of this substitution is:

$$\begin{aligned}
&l_{01}^2 l_{02}^2 l_{03}^2 (A_1A_2 + A_1A_3 + A_2A_3 - A_1A_2A_3) X_0^2 \\
&+ l_{01}^2 l_{12}^2 l_{13}^2 (A_1A_2 + A_1A_3 - A_2A_3 + A_1A_2A_3) X_1^2 \\
&+ l_{02}^2 l_{12}^2 l_{23}^2 (A_1A_2 + A_2A_3 - A_1A_3 + A_1A_2A_3) X_2^2 \\
&+ l_{03}^2 l_{13}^2 l_{23}^2 (A_1A_3 + A_2A_3 - A_1A_2 + A_1A_2A_3) X_3^2 = 0
\end{aligned}$$

Putting in the values of the A's, the result reduces to:

$$(43)\left\{\begin{aligned}
&l_{01}^2 l_{02}^2 l_{03}^2 (l_{13}^2 y_1 y_3 + l_{23}^2 y_2 y_3 + l_{12}^2 y_1 y_2) X_0^2 \\
&+ l_{01}^2 l_{12}^2 l_{13}^2 (l_{02}^2 y_0 y_2 + l_{03}^2 y_0 y_3 + l_{23}^2 y_2 y_3) X_1^2 \\
&+ l_{02}^2 l_{12}^2 l_{23}^2 (l_{01}^2 y_0 y_1 + l_{03}^2 y_0 y_3 + l_{13}^2 y_1 y_3) X_2^2 \\
&+ l_{03}^2 l_{13}^2 l_{23}^2 (l_{01}^2 y_0 y_1 + l_{02}^2 y_0 y_2 + l_{12}^2 y_1 y_2) X_3^2 = 0.
\end{aligned}\right.$$



This gives, at once, a relation between the points y and y', which are any two of the second set of 4 points. This relation is of such a nature that one y' determines 3 y's, for this point y' in S_y determines a point x, in S_x which in turn determines the 4 points of which this y' is one, or it determines the other 3 points of the 4.

as these 4 points form a tetrahedron which is self-polar with regard to the quadric, (43), 3 of the 4 points will lie on the polar plane of this fourth point. If the plane of the 3 points, y, is taken to be η, its coordinates will be:

$$\eta_0' = \alpha_{01}\alpha_{02}\alpha_{03}(\alpha_{13}y_1y_3+\alpha_{23}y_2y_3+\alpha_{12}y_1y_2)y_0$$
$$\eta_1' = \alpha_{01}\alpha_{12}\alpha_{13}(\alpha_{02}y_0y_2+\alpha_{03}y_0y_3+\alpha_{23}y_2y_3)y_1$$
$$\eta_2' = \alpha_{02}\alpha_{12}\alpha_{23}(\alpha_{01}y_0y_1+\alpha_{03}y_0y_3+\alpha_{13}y_1y_3)y_2$$
$$\eta_3' = \alpha_{03}\alpha_{13}\alpha_{23}(\alpha_{01}y_0y_1+\alpha_{02}y_0y_2+\alpha_{12}y_1y_2)y_3$$

This is a cubic transformation of the y's.

If we make y' incident with η, i.e. $(\eta y') = 0$, we get the equation of the Jacobian as given in (33), to within a factor. Also if in (43) we make x coincident with y, we get at once get the equation of the Jacobian of the net of quadrics.

Vita

Harry Clinton Gossard was born at Helena Ohio on March 13 1884. He was prepared for college in the public schools and graduated from Ohio Northern University with the degree of Bachelor of Science in July 1907. In October 1910, he entered the Johns Hopkins University, as a candidate for the degree of Doctor of Philosophy, with Mathematics as his principal subject and Astronomy and Education as first and second subordinates. He held a University Scholarship during the year 1911-1912 and a University Fellowship during the year 1912-1913.